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Abstract

摘要

We propose a nonlocal field theory for gravity in the presence of matter consistent with perturbative unitarity, quantum finiteness, and other essential classical properties that we are going to list below. First, the theory exactly reproduces the same tree-level scattering amplitudes of Einstein's gravity coupled to matter insuring no violation of macro-causality. Second, all the exact solutions of Einstein's theory are also exact solutions of the nonlocal theory. Finally, and most importantly, the linear and nonlinear stability analysis of the exact solutions in nonlocal gravity (with or without matter) is in one-to-one correspondence with the same analysis in General Relativity. Therefore, all the exact solutions stable in Einstein's theory are also stable in nonlocal gravity in the presence of matter at any perturbative order.

我们提出了一种含物质的非局域引力场论，该理论满足微扰幺正性、量子有限性以及下文将列出的其他基本经典性质。首先，该理论精确复现了爱因斯坦引力耦合物质的所有树级散射振幅，确保不破坏宏观因果性。其次，爱因斯坦理论的所有精确解也都是该非局域理论的精确解。最后且最重要的是，(含物质或不含物质的) 非局域引力中精确解的线性与非线性稳定性分析，与广义相对论中的对应分析一一对应。因此，所有在爱因斯坦理论中稳定的精确解，在含物质的非局域引力中任意微扰阶下也同样稳定。

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非局域场论 - 量子场论 - 超可重整化理论 - 幺正理论

Introduction

引言

Nonlocal field theory aims to provide a simple, compact, and elegant solution to the quantum gravity issue. Indeed, gravity at quantum level is not special but exactly like all the other fundamental interactions. However, if classical gravity is described by the Einstein-Hilbert action, then the outcome of the quantization procedure shows divergences that drastically change the structure of the theory. In the technical jargon of quantum field theory, it is said that Einstein's theory of gravity is non-renormalizable. However, there is not any inconsistency between gravity and quantum mechanics: Again, gravity is just non-renormalizable and need for a completion at high energy (a new action principle). This is something that does not happen in QED and QCD, but that physicists were called to face for the case of the Fermi theory of weak interactions. The renormalizability problem of the latter theory was overcome replacing Fermi's Lagrangian with a non-abelian gauge theory. Therefore, exactly like for the weak interactions, also for gravity we need a new gravitational theory able to tame the infinities. Now we know that the only possible extension of the Einstein-Hilbert theory, in the field theory framework, that is consistent with unitarity and renormalizability (actually finiteness to all perturbative orders in the loop expansion) is the weakly nonlocal gravity. Another possibility is provided by the Lee-Wick quantum gravity that, however, needs further prescriptions at classical and quantum levels, which are not specified by the action (references will be quoted shortly).

非局域场论旨在为量子引力问题提供一个简洁、紧凑且优雅的解决方案。事实上，量子层面的引力并不特殊，它和其他所有基本相互作用完全一样。但如果经典引力由爱因斯坦-希尔伯特作用量描述，那么量子化过程的结果会出现发散，彻底改变理论的结构。用量子场论的专业术语来说，爱因斯坦引力理论是不可重整化的。然而，引力和量子力学之间并不存在任何矛盾: 引力只是不可重整化，需要在高能量下完成完备化 (一个新的作用量原理)。这种情况不会出现在量子电动力学 (QED) 和量子色动力学 (QCD) 中，但物理学家在费米弱相互作用理论中就曾不得不面对这类问题。费米理论的可重整化问题通过将费米拉格朗日量替换为非阿贝尔规范理论得到解决。因此，和弱相互作用的情况完全一样，引力也需要一个新的引力理论来处理无穷大问题。现在我们知道，在场论框架下，爱因斯坦-希尔伯特理论唯一满足幺正性和可重整化 (实际上在圈展开的所有微扰阶上都是有限的) 的可能推广就是弱非局域引力。另一种可能由李-威克量子引力提供，但该理论在经典和量子层面都需要额外的规定，而这些规定并没有由作用量给出 (我们很快会列出参考文献)。

Records show that a nonlocal gravitational theory for gravity was proposed by Krasnikov [1] and studied 2 years later by Kuz'min [2]. However, only recently a multidimensional generalization of the theory [3], with particular attention to odd dimension [4] and an extension of the theory in even dimension [5], has been shown to be finite at quantum level. Furthermore, the Cutkosky rules [6] for a general nonlocal field theory have been derived in [7-9], where the perturbative unitarity was proven at any order in the loop expansion including the analysis of the anomalous thresholds. The macro-causality is also secured proven in [10, 11] on the base of Shapiro's time delay. On the other hand, the local Lee-Wick quantum gravity has been proposed in [12, 13] , on the footprint of the seminal paper [14], in order to address the unitarity issue that plagues local higher derivative theories.

有记录显示，克拉什尼科夫最早提出了非局域引力理论 [1]，库兹明在两年后对该理论展开了研究 [2]。但直到最近，该理论的多维推广 [3](尤其关注奇维 [4]) 以及偶维下该理论的扩展 [5] 才被证明在量子层面是有限的。此外，一般非局域场论的卡特科夫斯基规则已在文献 [7-9] 中推导得出，文中在包含反常阈值分析的情况下证明了圈展开任意阶的微扰幺正性。基于夏皮罗时间延迟，宏观因果性也在 [10, 11] 中得到了可靠证明。另一方面，局部李-威克量子引力是在开创性论文 [14] 的基础上，为解决困扰局域高阶导数理论的幺正性问题，在 [12, 13] 中被提出的。

Despite very encouraging, we would like to say surprising results listed above, not much has been done for gravity in the presence of matter [16, 17]. A simple way to couple nonlocal gravity to matter is introducing supersymmetry. This is a relatively easy task when we have at our disposal a superspace formalism [18], but the construction of other theories is still incomplete, see, for example, the eleven-dimensional supergravity [19].

尽管上述成果十分令人鼓舞，甚至可以说令人惊喜，但针对存在物质的引力的相关研究仍然很少 [16, 17]。将非局域引力与物质耦合的一种简单方法是引入超对称。如果我们拥有超空间形式体系 [18]，这项工作会相对容易，但其他理论的构建仍不完备，例如十一维超引力 [19]。

In this chapter we provide a recipe to construct a general nonlocal field theory for gravity coupled to matter (NLGM) on the base of the following four requirements (by Einstein's theory we will mean Einstein's gravity in the presence of matter):

在本章中，我们基于以下四个条件，给出了构建耦合物质的一般非局域引力场论 (NLGM) 的方案 (下文所称爱因斯坦理论均指存在物质的爱因斯坦引力):

(i) All the solutions of Einstein's gravity must be solutions of NLGM (this is an empirical requirement).

(i) 所有爱因斯坦引力的解都必须是 NLGM 的解 (这是经验性要求)。

(ii) All the tree-level scattering amplitudes of NLGM theory must coincide with those of Einstein's theory (this requirement guarantees macro-causality).

(ii) NLGM 理论的所有树级散射振幅必须与爱因斯坦理论的树级散射振幅一致 (该要求保证宏观因果性)。

(iii) The stability analysis of the exact solutions in NLGM has to be in one-toone correspondence with the same analysis in Einstein's theory (namely, if a solution is stable in Einstein's gravity, it is stable in NLGM too).

(iii) NLGM 中精确解的稳定性分析必须与爱因斯坦理论中的对应分析一一对应 (即如果一个解在爱因斯坦引力中稳定，那么它在 NLGM 中也稳定)。

(iv) The theory has to be super-renormalizable or finite at quantum level and unitary at any perturbative order in the loop expansion.

(iv) 该理论必须在量子层面是超可重整化或有限的，且在圈展开的任意微扰阶上都是幺正的。

As a final remark, we want to emphasize that the recipe provided in this chapter will allow us to construct the ultraviolet completion of any local two-derivative theory, in any dimension, and regardless of the presence of gravity.

最后我们需要强调，本章给出的方案可以让我们构造任意局域二导数理论的紫外完备化，适用于任意维度，且不受是否存在引力的限制。

Nonlocal Gravity-Matter Theory

非局域引力-物质理论

In this section we first display the general theory accomplish for requirements (i)-(iv) listed above in the previous section, and afterward, we will comment on such properties. Hence, let us start with the action

在本节中，我们首先展示满足上一节列出的要求 (i)-(iv) 的一般理论，随后我们会讨论该理论的相关性质。下面我们从作用量开始

$$S\left[\Phi_i\right] = \int \mathrm{d}^D x \sqrt{-g}\left(\mathcal{L}_\ell + E_i F(\Delta)_{ij} E_j\right), \tag{1}$$

$$S_\ell = \int \mathrm{d}^D x \sqrt{-g} \mathcal{L}_\ell, \ \mathcal{L}_\ell = \frac{2}{\kappa^2} R + \mathcal{L}_m, \tag{2}$$

$$E_i\left(x\right) = \frac{\delta S_\ell}{\delta \Phi_i\left(x\right)}, \tag{3}$$

where Φ_i is a set of fields, placed in a vector of components labelled by the index i , that include the metric and the matter’ s fields. $F(\Delta)_{ij}\left(x, y\right)$ is a symmetric (with respect to the swap of the indexes i and j together with the spacetime points x and y) tensorial entire function whose argument is a tensorial differential operator Δ that we are going to construct consistently with the stability of the exact solutions of the local theory (namely solutions of the equations of motion (EoM) $E_i = 0$). Indeed, it is straightforward to show that the requirement (i) is satisfied by explicitly computing the variation of the action (1) (up to the total derivative terms and operators quadratic in the EoM E_i). The EoM for the nonlocal action (3) (see Appendix ”Appendix A: Equations of Motion” for more details) reads

其中 Φ_i 是一组场，整理为分量由指标 i 标记的矢量，包含度规和物质场。$F(\Delta)_{ij}\left(x, y\right)$ 是一个对称整函数张量 (对指标 i 与 j 交换、时空点 x 与 y 交换都对称)，其自变量是张量微分算子 Δ ，我们将构造该算子使之与局域理论精确解的稳定性相容 (即运动方程 (EoM) 的解 $E_i = 0$)。事实上，我们可以直接证明: 通过显式计算作用量 (1) 的变分 (不计全导数项和运动方程的二次算符项 E_i)，要求 (i) 是满足的。非局域作用量 (3) 的运动方程 (更多细节参见附录“附录 A: 运动方程”) 为

$$\mathcal{E}_k = E_k + 2\Delta_{ki} F_{ij} E_j + O\left(E^2\right) = 0. \tag{4}$$

Since E_k are Einstein’ s EoM and the EoM for the matter, the following implication applies:

由于 E_k 是爱因斯坦运动方程和物质的运动方程，因此可以得到:

$$E_k = 0 \;\Rightarrow\; \mathcal{E}_k = 0, \tag{5}$$

where we introduced the Hessian operator of the local theory defined by (see Appendix "Appendix A: Equations of Motion") (For the reader who needs to see the explicit form of the Hessian, we refer to the paper [23], where the components of Δ are provided for gravity coupled to a complex scalar, a dirac fermion, and an abelian gauge vector.):

> 此处我们引入局域理论的黑塞算子，定义如下 (参见附录“附录 A: 运动方程”。若读者需要查看黑塞的显式形式，可参考文献 [23]，其中给出了引力耦合复标量、狄拉克费米子和阿贝尔规范矢量时 Δ 的分量):

$$\Delta_{ki} \equiv \frac{\delta E_i}{\delta \Phi_k} = \frac{\delta^2 S_\ell}{\delta \Phi_k \delta \Phi_i}. \tag{6}$$

The same property, namely that the action consists on the Lagrangian in (2) plus a second operator quadratic in E_i , secures that all the scattering amplitudes of the nonlocal theory in the presence of matter are identical to those of Einstein's gravity coupled to matter. The latter statement is based on a simple generalization of the theorem already used in [10, 11, 20] . Therefore, also the requirement (ii) is satisfied. Notice that the reverse implication in (5) is not true because the space of solutions of the nonlocal theory is generally larger than the one of the local Einstein theory coupled to matter.

> 同样的性质，即作用量由 (2) 中的拉格朗日量加上一个以 E_i 为二次项的算子构成，保证了非局域理论在存在物质时所有散射振幅都和爱因斯坦引力耦合物质的散射振幅完全一致。该结论基于对 [10, 11, 20] 中已使用定理的简单推广。因此要求 (ii) 也得到满足。注意 (5) 中的逆命题不成立，因为非局域理论的解空间通常比局域爱因斯坦引力耦合物质的解空间更大。

Notice that the action (1) is very general and the recipe defined by the equations (1), (2), and (3) applies to any system in any dimension, starting from a 1 +0 system to a D -dimensional field theory. Therefore,(1) is actually an ultraviolet completion of any field theory, including a point-like action, with an arbitrary number of fields in the presence or the absence of the gravitational interaction.

> 注意作用量 (1) 非常普适，由方程 (1)、(2) 和 (3) 定义的构造方法适用于任意维度的任意系统，从 1+0 维系统到 D 维场论都成立。因此 (1) 实际上是任意场论 (包括点作用量，任意数量场、存在或不存在引力相互作用) 的紫外完备化。

In order to address the stability issue as stated in (iii), we have to pick out a form factor F_{ij} satisfying the following equation (For dimensional reasons, the operator Δ , being the argument of the entire function H , has to be divided by a proper power of the mass scale Λ . For example, in $D = 4$, if we are differentiating Einstein's EoM with respect to the metric, we should divide by Λ^4 , if we differentiate Einstein's EoM with respect to a scalar field or the matter EoM with respect to the metric in both cases, we should divide by Λ^3 , and finally, if we differentiate the matter EoM with respect to the scalar field, we have to divide by Λ^2 .),

为了解决 (iii) 中提出的稳定性问题，我们需要选取一个满足以下方程的形状因子 F_{ij} (量纲分析说明，作为整函数 H 自变量的算子 Δ 必须除以质量标度 Λ 的适当次幂。例如在 $D=4$ 中，如果我们对爱因斯坦运动方程关于度规求导，应除以 Λ^4；如果我们对爱因斯坦运动方程关于标量场求导，或对物质运动方程关于度规求导，两种情况都应除以 Λ^3；最后，如果我们对物质运动方程关于标量场求导，则应除以 Λ^2),

$$2\Delta_{ik}F(\Delta)_{kj} \equiv \left(e^{H(\Delta_\Lambda)} - 1\right)_{ij}, \tag{7}$$

where $H(\Delta)$ is an entire analytic function (see Appendix "Appendix A: Equations of Motion" for the explicit construction of $F(\Delta)$). Indeed, replacing (7) in (4), the EoM turns into

其中 $H(\Delta)$ 是整解析函数 ($F(\Delta)$ 的显式构造参见附录“附录 A: 运动方程”)。将 (7) 代入 (4) 后，运动方程变为

$$\mathcal{E}_k = \left(e^{H(\Delta_\Lambda)}\right)_{kj} E_j + O\left(E^2\right) = 0. \tag{8}$$

Since the function in front of the local EoM E_j is invertible, we can infer that the theory is ghost-free and only the perturbative degrees of freedom of Einstein's gravity in the presence of matter are allowed to propagate. As marked by $O\left(E^2\right)$ in the EoM (8) and consistently with the previous results published in [21,22], the EoM for the perturbations of the metric and all the other fields of the theory are the same as in Einstein's local gravity with matter. Hence, the stability is guaranteed at any perturbative order. Let us expand on this statement. We can invert the exponential factor and rewrite (8) as

由于局域运动方程 E_j 前的函数可逆，我们可以推断该理论无鬼，只有存在物质时爱因斯坦引力的微扰自由度可以传播。正如运动方程 (8) 中 $O\left(E^2\right)$ 所标注，且与已发表在 [21,22] 的先前结果一致，该理论中度量扰动和所有其他场的运动方程，与有物质的爱因斯坦局域引力中的运动方程完全相同。因此，任意微扰阶的稳定性都可以得到保证。我们来进一步展开这个结论。我们可以对指数因子求逆，将 (8) 重写为

$$\widetilde{\mathcal{E}}_i \equiv E_i + \left(e^{-H(\Delta_\Lambda)}\right)_{ik}\left[O\left(E^2\right)\right]_k = 0. \tag{9}$$

Now, given an exact background solution of the NLGM theory compatible with $E_k = 0$, we can derive the EoM for the perturbations defined through an expansion of the fields, and then of the EoM, in a small dimensionless parameter ε , namely

现在，已知 NLGM 理论中一个满足 $E_k = 0$ 的精确背景解，我们可以推导通过场展开定义的扰动的运动方程，随后对运动方程本身做展开，展开围绕小无量纲参数 ε 进行，即

$$\Phi_i = \sum_{n=0}^{\infty} \varepsilon^n \Phi_i^{(n)} \tag{10}$$

$$E_k\left(\Phi_i\right) = \sum_{n=0}^{\infty} \varepsilon^n E_k^{(n)}, \ \widetilde{\mathcal{E}}_k\left(\Phi_i\right) = \sum_{n=0}^{\infty} \varepsilon^n \widetilde{\mathcal{E}}_k^{(n)}. \tag{11}$$

Assuming that the fields $\Phi_i^{(0)}$ satisfy the local background EoM, namely

假设场 $\Phi_i^{(0)}$ 满足局域背景运动方程，即

$$E_k^{(0)}\left(\Phi_i^{(0)}\right)=0, \tag{12}$$

it is extremely simple to prove the following theorem, which is a slight generalization of the theorems proved in [21,22].

我们可以非常简单地证明以下定理，它是对 [21,22] 中已证明定理的小幅推广。

Theorem. In the NLGM theory, all perturbations (for gravity and matter) satisfy the same EOM of the perturbations in Einstein's gravity coupled to matter, namely

定理: 在 NLGM 理论中，所有扰动 (引力扰动和物质扰动) 都满足与耦合物质的爱因斯坦引力中扰动相同的运动方程，即

$$\widetilde{\mathcal{E}}_k^{(n)}\left(\Phi_i^{(n)}\right)=0 \;\Rightarrow\; E_k^{(n)}\left(\Phi_i^{(n)}\right)=0 \text{ for } n>0, \tag{13}$$

where the label " n " stays for the perturbative expansion of the tensors $\widetilde{\mathcal{E}}_k$ and the EoM E_k at the order " n " in all the perturbations $\Phi_i^{(n)}$.

其中标记" n " 代表张量 $\widetilde{\mathcal{E}}_k$ 和运动方程 E_k 在所有扰动 $\Phi_i^{(n)}$ 的 n 阶的微扰展开。

The proof is a straightforward consequence of the EoM (9), which coincides with Einstein' s $E_k=0$ EoM in the presence of matter up to operators $O\left(E^2\right)$, and of the invertibility of the exponential form factor. The perturbative expansion that provides the details of the proof is identical to the one for pure gravity. Therefore, we remind the reader to [22], but a short proof is given in Appendix "Appendix D: Proof of the Theorem in the Main Text". In particular, it deserves to be noticed that in the perturbative expansion in ε of equation (8) the exponential form factor always contributes at the zero order, namely ε^0 . Moreover, for the standard model of particle physics, the Hessian resulting from the local action is diagonal (or constant) at the order ε^0 (see [23]). Hence, the inversion (9) is actually trivial at any perturbative order.

该证明是运动方程 (9) 的直接推论: 方程 (9) 在算符 $O\left(E^2\right)$ 范围内与有物质存在时的爱因斯坦 $E_k=0$ 运动方程一致，且指数形状因子可逆。给出证明细节的微扰展开与纯引力的微扰展开完全相同，因此我们请读者参考文献 [22]，不过本文附录“附录 D: 正文定理的证明”中也给出了简要证明。需要特别注意的是，在对方程 (8) 做 ε 的微扰展开时，指数形状因子始终贡献零阶，即 ε^0 。此外，对于粒子物理标准模型，局域作用量导出的黑塞矩阵在 ε^0 阶是对角的 (或常数的)(参见文献 [23])，因此在任意微扰阶，求逆操作 (9) 实际上都是平凡的。

The Quantum Theory

量子理论

Making use of the previous results, we show perturbative unitarity and finiteness of the theory (1) with form factor (7) and Hessian operator given in (6) and (14).

利用前文所得结果，我们证明了带有形状因子 (7)、黑塞算子由 (6) 和 (14) 给出的理论 (1) 满足微扰幺正性且有限。

In order to show the perturbative unitarity of the NLGM theory, we do not need to compute the propagator because the EoM already tells us that we have only one pole in $k^2 = -m_i^2$ for each field, exactly like in the local theory (see Appendix "Appendix B: Propagators"). Therefore, the singularities of the amplitudes are obtained from Landau's equations as in the local case (this is due to the analytic structure of the form factor F_{ij}) and the Cutkosky rules are the same of the local theory, as well. Hence, we can export the outcome of [7] to the general theory presented in this chapter, and the unitarity is guaranteed at any perturbative order in the loop expansion. We remind that the theory has to be defined in the Euclidean space and the physical amplitudes are afterward obtained employing the analytic continuation from complex to real external energies. The last comment is about the non-diagonal elements of the operator Δ . Indeed, one can easily see that such components are at least linear in the fields [23] and, therefore, cannot affect the propagators around the Minkowski spacetime.

为了证明非局域引力-物质理论 (NLGM) 的微扰幺正性，我们无需计算传播子，因为运动方程已经说明，每个场在 $k^2 = -m_i^2$ 中仅存在一个极点，这与局域理论完全一致 (参见附录“附录 B: 传播子”)。因此，振幅的奇点可由朗道方程得到，与局域情形相同 (这源于形状因子 F_{ij} 的解析结构)，且卡特斯基规则也与局域理论完全一致。因此，我们可以将文献 [7] 的结论推广到本章提出的广义理论，圈展开中任意微扰阶的幺正性都得到保证。需要提醒的是，该理论必须定义在欧几里得空间中，之后通过从复能量到实外部能量的解析延拓得到物理振幅。最后讨论算子 Δ 的非对角元: 不难发现，这些分量至少是场的线性项 [23]，因此不会影响闵可夫斯基时空背景下的传播子。

We can finally address the issue of the quantum divergences. The differential operator Δ in (14) is a second-order differential operator, and hence, it is always possible to choose a form factor $\exp H(\Delta_\Lambda)$, asymptotically polynomial [3-5], to cancel all the divergences from two loops onward. However, in even dimension we still have one-loop divergences that can likely be removed by adding other operators to the action (1) provided they are at least cubic in the equations of motion E_k , for example, in $D = 4$ a viable operator could be $E^2 \Delta^{\gamma-2} E^2$ (the indexes must be properly contracted), where $\gamma > 2$ is an integer. The super-renormalizability requires the form factor to scale at high energy like $F(\Delta) \to \Delta^\gamma$, while the propagator scales like $1/k^{2\gamma+4}$.

我们最终可以讨论量子发散问题。(14) 中的微分算子 Δ 是二阶微分算子，因此总能选取一个渐近多项式形式的形状因子 $\exp H(\Delta_\Lambda)$ [3-5]，抵消从两圈开始的所有发散。但在偶数维空间中，我们仍然存在一圈发散，只要在运动方程 E_k 中添加至少三次方的其他算子，通常就可以消除这些发散，例如在 $D = 4$ 中一个可行的算子是 $E^2 \Delta^{\gamma-2} E^2$ (指标需要适当缩并)，其中 $\gamma > 2$ 是整数。超可重整化性要求形状因子在高能量下按 $F(\Delta) \to \Delta^\gamma$ 标度，而传播子按 $1/k^{2\gamma+4}$ 标度。

For the sake of simplicity, we can consider the NLGM theory in odd dimension where we do not have one-loop divergences in the dimensional regularization scheme. Therefore, the theory proposed in this chapter is surely finite in odd dimensions and super-renormalizable in even dimensions. The latter property ensures that all the interactions described by the action (1) are asymptotically free in the ultraviolet regime [25]. In odd dimension or even dimension if we add several local operators, the theory is finite and then conformal invariant [4, 5]. Both scenarios presented above are perfectly consistent with the unitarity bound.

为简化起见，我们可以讨论奇数维空间中的 NLGM 理论，在维数正规化方案中奇数维不存在一圈发散。因此，本章提出的理论在奇数维中必然有限，在偶数维中是超可重整化的。后一性质保证了作用量 (1) 描述的所有相互作用在紫外区都是渐近自由的 [25]。在奇数维，或添加了多个局域算子的偶数维中，该理论是有限的，因此具有共形不变性 [4, 5]。上述两种情景都与幺正性界完全自洽。

Conclusions

结论

We have explicitly constructed a nonlocal theory for all fundamental interactions, including gravity, that at classical level has the same solutions and the same stability properties of the local Einstein theory in the presence of matter. Moreover, the theory reproduces all and only the same tree-level scattering amplitudes of the local standard model of particle physics in the presence of gravity, securing that there is no causality violation [10,25]. At quantum level, the theory is unitary and surely finite in odd dimension, while in even dimension there are only one-loop divergences that can be removed adding few more local operators on the footprint of what has been done for pure gravity without and with the cosmological constant [5, 20].

我们已经明确构建了涵盖引力在内所有基本相互作用的非局部理论，该理论在经典层面，存在物质时的解与稳定性性质均和局部爱因斯坦理论一致。此外，该理论精确复现了存在引力时粒子物理局部标准模型的全部且仅那些树级散射振幅，确保不存在因果性破坏 [10,25]。在量子层面，该理论具有幺正性，在奇维空间中确定是有限的；而在偶维空间中仅存在单圈发散，按照处理不带宇宙常数和带宇宙常数的纯引力时的思路，只需添加少量额外局部算符即可消除这些发散 [5, 20]。

Thus, this chapter lays strong foundations for an ultraviolet completion of the standard model of particle physics and gravity.

因此，本章为粒子物理标准模型和引力的紫外完备化奠定了坚实基础。

Appendix A: Equations of Motion

附录 A: 运动方程

In this section we derive the EoM for a general nonlocal theory providing all the details of the calculation. We will make use of the following definition:

本节我们将推导一般非局域理论的运动方程，并给出计算的全部细节。我们会用到以下定义:

$$\Delta_{ki}(y,x) \equiv \frac{\delta E_i(x)}{\delta \Phi_k(y)} = \frac{\delta^2 S_\ell}{\delta \Phi_k(y)\,\delta \Phi_i(x)} = \Delta_{ki}(y)\,\frac{\delta^D(y-x)}{\sqrt{-g(x)}}, \tag{14}$$

and we will use the functional derivative consistent with the Dirac delta distribution in curved spacetime

且我们将使用与弯曲时空狄拉克 δ 分布一致的泛函导数

$$\frac{\delta\Phi_i(x)}{\delta\Phi_j(y)} = \frac{\delta^D(x-y)}{\sqrt{-g(y)}}\delta_{ij}. \tag{15}$$

The variation of the action, introducing the short notation for the integral measure $\int d\mu(x) \equiv \int d^D x\sqrt{-g(x)}$ or simply $\int_x$, reads

对作用量变分，引入积分测度的简写 $\int d\mu(x) \equiv \int d^D x\sqrt{-g(x)}$，或简记为 $\int_x$，结果为

$$\begin{aligned}
\delta S = &\int \mathrm{d}\mu(x)\left(\delta\Phi_i E_i + \delta E_i F_{ij} E_j + E_i F_{ij}\delta E_j + O\left(E^2\right)\right), \\
= &\int \mathrm{d}\mu(x)\Big[\delta\Phi_i(x) E_i(x) + \int \mathrm{d}\mu(y)\Big(\frac{\delta E_i(x)}{\delta\Phi_k(y)}\delta\Phi_k(y) F_{ij}(x) E_j(x) \\
&+ E_i(x) F_{ij}(x)\frac{\delta E_j(x)}{\delta\Phi_k(y)}\delta\Phi_k(y) + O\left(E^2\right)\Big)\Big] \\
= &\int \mathrm{d}\mu(x)\Big[\delta\Phi_k(x) E_k(x) + \int \mathrm{d}\mu(y)\Big(\frac{\delta E_i(x)}{\delta\Phi_k(y)}\delta\Phi_k(y) F_{ij}(x) E_j(x) \\
&+ E_j(x) F_{ji}(x)\frac{\delta E_i(x)}{\delta\Phi_k(y)}\delta\Phi_k(y) + O\left(E^2\right)\Big)\Big],
\end{aligned} \tag{16}$$

where in the last term we just changed name to the indexes in order to have the same Δ operator (see (14)) of the last by one term. Let us now introduce the following definition:

其中在最后一项中我们仅改写了指标名称，使其与倒数第二项拥有相同的 Δ 算符 (见式 (14))。现在我们引入以下定义:

$$\begin{aligned}
v_i(x) = &\int \mathrm{d}^D y\sqrt{-g(y)}\Delta_{ki}(y,x)\,\delta\Phi_k(y) \equiv \int \mathrm{d}\mu(y)\,\Delta_{ki}(y,x)\,\delta\Phi_k(y) \\
= &\int \mathrm{d}\mu(y)\,\delta\Phi_k(y)\,\Delta_{ki}(y,x).
\end{aligned} \tag{17}$$

Therefore, replacing the definitions (14) and (17) in (16), we get

因此，将定义 (14) 和 (17) 代入 (16)，我们得到

$$\begin{aligned}
\delta S = &\int \mathrm{d}\mu(x)\Bigg[\delta\Phi_k(x) E_k(x) + \underbrace{\int \mathrm{d}\mu(y)\left(\Delta_{ki}(y,x)\,\delta\Phi_k(y)\right.}_{v_i(x)} F_{ij}(x) E_j(x) \\
&+ E_j(x) F_{ji}(x)\,\Delta_{ki}(y,x)\,\delta\Phi_k(y) + O\left(E^2\right))\Big] \\
= &\int \mathrm{d}\mu(x)\left[\delta\Phi_k(x) E_k(x) + v_i(x) F_{ij}(x) E_j(x) + E_j(x) F_{ji}(x) v_i(x) + O\left(E^2\right)\right] \\
= &\int \mathrm{d}\mu(x)\,\delta\Phi_k(x) E_k(x) + \int \mathrm{d}\mu(x)\int \mathrm{d}\mu(y)\left(v_i(x) F_{ij}(x,y) E_j(y)\right.
\end{aligned}$$

$$+E_j(x)F_{ji}(x,y)v_i(y)\big)+O\left(E^2\right). \tag{18}$$

In the last term we now change the name to the integration variables, namely $x \to y$ and $y \to x$, hence

现在我们对最后一项改写积分变量，即 $x \to y$ 和 $y \to x$ ， 由此可得

$$\delta S = \int \mathrm{d}\mu(x)\,\delta\Phi_k(x)\,E_k(x) + \int \mathrm{d}\mu(x)\int \mathrm{d}\mu(y)\big(v_i(x)\,F_{ij}(x,y)\,E_j(y)$$

$$+E_j(y)\,F_{ji}(y,x)\,v_i(x)\big)+O\left(E^2\right). \tag{19}$$

Making use of the following integrated symmetric property of the Hessian Δ (see the appendix "(Appendix C: Symmetry Properties of the Hessian)"), namely

利用黑塞矩阵 Δ 的如下积分对称性质 (见附录“附录 C: 黑塞矩阵的对称性质”)， 即

$$\int d^Dx\sqrt{-g(x)}\int d^Dy\sqrt{-g(y)}A_i(x)\,\Delta_{ij}(x,y)\,B_j(y)$$

$$=\int d^Dx\sqrt{-g(x)}\int d^Dy\sqrt{-g(y)}B_j(y)\,\Delta_{ji}(y,x)\,A_i(x), \tag{20}$$

the variation turns into

变分可化为

$$\delta S = \int \mathrm{d}\mu(x)\,\delta\Phi_k(x)\,E_k(x) + \int \mathrm{d}\mu(x)\int \mathrm{d}\mu(y)\big(v_i(x)\,F_{ij}(x,y)\,E_j(y)$$

$$+v_i(x)\,F_{ij}(x,y)\,E_j(y)\big)+O\left(E^2\right)$$

$$=\int \mathrm{d}\mu(x)\,\delta\Phi_k(x)\,E_k(x) + \int \mathrm{d}\mu(x)\int \mathrm{d}\mu(y)\,2v_i(x)\,F_{ij}(x,y)\,E_j(y)+O\left(E^2\right)$$

$$=\int \mathrm{d}\mu(x)\,[\delta\Phi_k(x)\,E_k(x)$$

$$+\underbrace{\int \mathrm{d}^Dy\sqrt{-g(y)}\delta\Phi_k(y)\,\Delta_{ki}(y,x)}_{v_i(x)}\int \mathrm{d}^Dz\sqrt{-g(z)}2F_{ij}(x,z)\,E_j(z)+\,O\left(E^2\right)]$$

$$=\int \mathrm{d}\mu(x)\left[\delta\Phi_k(x)\,E_k(x)+\int \mathrm{d}^Dy\sqrt{-g(y)}\left(\delta\Phi_k(y)\,\Delta_{ki}(y)\,\frac{\delta^D(y-x)}{\sqrt{-g(x)}}\right)\right.$$

$$\times\int \mathrm{d}^Dz\sqrt{-g(z)}2F_{ij}(x,z)\,E_j(z)+O\left(E^2\right)].$$

(21)

The last equation can be integrated by parts in the y variable, and we get

对最后一个方程关于 y 变量分部积分，我们得到

$$
\begin{aligned}
\delta S &= \int \mathrm{d}\mu(x)\left[\delta\Phi_k(x)E_k(x) + \int \mathrm{d}^D y\sqrt{-g(y)}\left(\delta\Phi_k(y)\Delta_{ki}(y)\frac{\delta^D(y-x)}{\sqrt{-g(x)}}\right)\right.\\
&\quad \times \int \mathrm{d}^D z\sqrt{-g(z)}2F_{ij}(x,z)E_j(z) + O\left(E^2\right)]\\
&= \int \mathrm{d}\mu(x)\left[\delta\Phi_k(x)E_k(x) + \int \mathrm{d}^D y\Delta_{ki}(y)\left(\sqrt{-g(y)}\delta\Phi_k(y)\right)\frac{\delta^D(y-x)}{\sqrt{-g(x)}}\right.\\
&\quad \left.\times \int \mathrm{d}^D z\sqrt{-g(z)}2F_{ij}(x,z)E_j(z) + O\left(E^2\right)\right]\\
&= \int \mathrm{d}\mu(x)\left[\delta\Phi_k(x)E_k(x) + \Delta_{ki}(x)\left(\sqrt{-g(x)}\delta\Phi_k(x)\right)\frac{1}{\sqrt{-g(x)}}\right.\\
&\quad \times \int \mathrm{d}^D z\sqrt{-g(z)}\,2F_{ij}(x,z)E_j(z) + O\left(E^2\right)]\\
&= \int \mathrm{d}\mu(x)\left[\delta\Phi_k(x)E_k(x) + \sqrt{-g(x)}(\Delta_{ki}(x)\,\delta\Phi_k(x))\frac{1}{\sqrt{-g(x)}}\right.\\
&\quad \times \int \mathrm{d}^D z\sqrt{-g(z)}\,2F_{ij}(x,z)E_j(z) + O\left(E^2\right)]\\
&= \int \mathrm{d}\mu(x)\,[\delta\Phi_k(x)E_k(x) + (\Delta_{ki}(x)\,\delta\Phi_k(x))\\
&\quad \times \int \mathrm{d}^D z\sqrt{-g(z)}\,2F_{ij}(x,z)E_j(z) + O\left(E^2\right)\Big]\,.
\end{aligned}
\tag{22}
$$

Notice that the covariant derivative of the determinant is zero because the metric is connection-compatible, and therefore, $\sqrt{-g(x)}$ can be moved on the left side of the Hessian in the last but one step.

注意，行列式的协变导数为零，因为度规与联络兼容，因此在倒数第二步中可将 $\sqrt{-g(x)}$ 移到黑塞矩阵的左侧。

Integrating now by parts in x ,

现在对 x 做分部积分,

$$
\begin{aligned}
\delta S &= \int \mathrm{d}\mu(x)\,[\delta\Phi_k(x)E_k(x)\\
&+\delta\Phi_k(x)\left(\Delta_{ki}(x)\int \mathrm{d}^D z\sqrt{-g(z)}2F_{ij}(x,z)E_j(z)\right) + O\left(E^2\right)].
\end{aligned}
\tag{23}
$$

It deserves to be noticed that the operators

值得注意的是，这些算符

$$v_i(x),\ \Delta_{ij}(x,y),\ F_{ij}(x,y),\ E_j(y),\ \delta\Phi_k(y) \tag{24}$$

can be freely interchanged because each of them is in a closed integral form; namely they are not differential operators acting on their right or left side, but they are actually integrated quantities in which differential operators, if any, act on Dirac's delta distributions. This property has been used, for example, in the last equality in (17).

可以自由交换顺序，因为每个算符都处于闭合积分形式中；即它们并非作用在自身左右侧的微分算符，而是实际的积分量，其中即使存在微分算符，也仅作用于狄拉克 δ 分布。例如式 (17) 的最后一个等式就用到了这一性质。

In a more compact and implicit notation, the result (23) turns into

采用更紧凑的隐式记法，结果 (23) 可写为

$$\begin{aligned}\delta S &= \int \mathrm{d}^D x\sqrt{-g(x)}\left[\delta\Phi_k(x)E_k(x) + \delta\Phi_k(x)\,2\Delta_{ki}(x)F_{ij}(x)E_j(x) + O\left(E^2\right)\right]\\ &= \int \mathrm{d}^D x\sqrt{-g(x)}\delta\Phi_k(x)\left[E_k(x) + 2\Delta_{ki}(x)F_{ij}(x)E_j(x) + O\left(E^2\right)\right],\end{aligned} \tag{25}$$

and

且

$$\begin{aligned}\mathcal{E}_l(y) = \frac{\delta S}{\delta\Phi_l(y)} &= \int \mathrm{d}^D x\sqrt{-g(x)}\frac{\delta\Phi_k(x)}{\delta\Phi_l(y)}\left[E_k(x) + 2\Delta_{ki}(x)F_{ij}(x)E_j(x) + O\left(E^2\right)\right]\\ &= E_l(x) + 2\Delta_{li}(x)F_{ij}(x)E_j(x) + O\left(E^2\right),\end{aligned} \tag{26}$$

where we used (15). Finally, the EoM for the nonlocal theory reads

此处我们使用了式 (15)。最终，非局部理论的运动方程为

$$\mathcal{E}_k = E_k + 2\Delta_{ki}F_{ij}E_j + O\left(E^2\right) = 0, \tag{27}$$

and, making again explicit the dependence on spacetime points, the EoM (27) should be written as

若再次明确写出对时空点的依赖，运动方程 (27) 应写为

$$\mathcal{E}_k(x) = E_k(x) + \int d\mu(y)\int d\mu(z)\,2\Delta_{ki}(x,y)F_{ij}(y,z)E_j(z) + O\left(E^2\right) = 0. \tag{28}$$

In order to further simplify the above equation of motion (28) and get rid out of the instabilities, we now expand on the operator F_{ij} defined in (7). The analytic form factor F_{ij} is given as a solution of the following equation:

为进一步化简上述运动方程 (28) 并消除不稳定性，我们现在对式 (7) 中定义的算符 F_{ij} 展开分析。解析形状因子 F_{ij} 是以下方程的解:

$$2\Delta_{ik}F_{kj}(\Delta) = 2F_{ik}(\Delta)\Delta_{kj} = \left(e^{H}\right)_{ij} - 1_{ij}. \tag{29}$$

Indeed, if we define

事实上，若我们定义

$$F_{ij}(\Delta) = \sum_{n=0}^{+\infty} a_n(\Delta^n)_{ij} \ \text{ and } \ \left(e^{H(\Delta)}\right)_{ij} = \sum_{n=0}^{+\infty} b_n(\Delta^n)_{ij}, \tag{30}$$

by replacing (30) in (29), we get

将 (30) 代入 (29)，可得

$$2\Delta \sum_{n=0}^{+\infty} a_n\Delta^n = \sum_{n=0}^{+\infty} b_n\Delta^n - 1$$

$2\Delta \sum_{n=0}^{+\infty} a_n\Delta^n = b_0 + \sum_{n=1}^{+\infty} b_n\Delta^n - 1$ and assuming $b_0 = 1$ or $H(0) = 0$,

$2\Delta \sum_{n=0}^{+\infty} a_n\Delta^n = b_0 + \sum_{n=1}^{+\infty} b_n\Delta^n - 1$ ，并假设 $b_0 = 1$ 或 $H(0) = 0$,

$$\begin{aligned} 2\Delta \sum_{n=0}^{+\infty} a_n\Delta^n &= \sum_{n=1}^{+\infty} b_n\Delta^n \\ 2\Delta \sum_{n=0}^{+\infty} a_n\Delta^n &= \Delta \sum_{n=1}^{+\infty} b_n\Delta^{n-1} \ (n-1=k) \\ 2\Delta \sum_{n=0}^{+\infty} a_n\Delta^n &= \Delta \sum_{k=0}^{+\infty} b_{k+1}\Delta^k \end{aligned} \tag{31}$$

Between the third last and the last but one step in (31) we do not need to define the inverse of Δ because in the sum on the right side we have $n > 0$. Therefore, comparing the left and right sides of the last equality in (31), we figure out the relation between the coefficients a_n and b_n , namely

在式 (31) 的倒数第三步与倒数第一步之间，我们无需定义 Δ 的逆，因为右侧求和中已有 $n > 0$ 。因此，对比 (31) 中最后一个等式左右两侧，我们得到系数 a_n 与 b_n 的关系，即

$$a_n = \frac{b_{n+1}}{2} \tag{32}$$

Replacing (29) in (28),

将 (29) 代入 (28)，得

$$\mathcal{E}_k(x) = E_k(x) + \int d\mu(z)\left(e^{H(\Delta)} - 1\right)_{kj}(x,z)\, E_j(z) + O\left(E^2\right) = 0. \tag{33}$$

where the functional identity in (33) is defined by

其中式 (33) 中的泛函恒等式定义为

$$1_{kj}(x,z) = \delta_{kj}\frac{\delta(x-z)}{\sqrt{-g(z)}}. \tag{34}$$

Therefore, (33) turns into

因此，(33) 变为

$$\mathcal{E}_k(x) = E_k(x) + \int d\mu(z)\left[\left(e^{H(\Delta)}\right)_{kj}(x,z) - \delta_{kj}\frac{\delta(x-z)}{\sqrt{-g(z)}}\right] E_j(z) + O\left(E^2\right) = 0$$

$$\mathcal{E}_k(x) = E_k(x) + \int d\mu(z)\left(e^{H(\Delta)}\right)_{kj}(x,z)\, E_j(z)$$

$$- \int d\mu(z)\,\delta_{kj}\frac{\delta(x-z)}{\sqrt{-g(z)}} E_j(z) + O\left(E^2\right) = 0$$

$$\mathcal{E}_k(x) = E_k(x) + \int d\mu(z)\left(e^{H(\Delta)}\right)_{kj}(x,z)\, E_j(z) - E_k(x) + O\left(E^2\right) = 0$$

$$\mathcal{E}_k(x) = \int d\mu(z)\left(e^{H(\Delta)}\right)_{kj}(x,z)\, E_j(z) + O\left(E^2\right) = 0. \tag{35}$$

Now, using the second identity in (14), we end up with

现在，利用 (14) 中的第二个恒等式，我们最终得到

$$\mathcal{E}_k(x) = \int d\mu(z)\left[\left(e^{H(\Delta)}\right)_{kj}(x)\,\frac{\delta(x,z)}{\sqrt{-g(z)}}\right] E_j(z) + O\left(E^2\right) = 0$$

$$\mathcal{E}_k(x) = \left(e^{H(\Delta_x)}\right)_{kj} E_j(x) + O\left(E^2\right) = 0. \tag{36}$$

Equations (7) and (30) with (32) allow us to avoid to invert the Δ operator. Indeed, it deserves to be noticed that the definition of Δ^{-1} is extremely delicate. The Hessian Δ is usually not invertible because of gauge invariance (Also the operator $\Box$ is not invertible in flat space because of the zero mode/s, but not because of gauge invariance.), and one usually adds a gauge fixing term to the (local) action in order to get the inverse. If we define

式 (7)、(30) 与 (32) 使我们无需对 Δ 算符求逆。事实上，值得注意的是，Δ^{-1} 的定义极为精细。由于规范不变性，黑塞算符 Δ 通常不可逆 (算符 $\Box$ 在平直空间中也不可逆，原因是零模而非规范不变性)，通常人们会在 (局部) 作用量中添加规范固定项来得到逆算符。若我们定义

$$F_{ij} \equiv \left(\frac{e^{H(\Delta_\Lambda)} - 1}{2\Delta}\right)_{ij} \equiv \frac{1}{2}\left[\Delta^{-1}\left(e^{H(\Delta_\Lambda)} - 1\right)\right]_{ij}, \tag{37}$$

instead of (7) or (29), then Δ^{-1} will be part of the definition of the theory and one might worry about an explicit break of the gauge invariance of the theory because of the gauge fixing term. However, it will not be the case because Δ^{-1} would appear in the intermedium steps of our derivation, but not in the final action and in the EoM. Indeed, the function F_{ij} is analytic in Δ , and the presence of Δ^{-1} in (37) is just formal. If we wanted to be mathematically rigorous, we should add a gauge fixing term to Δ in (37); namely, we make the following replacement in (37):

而非 (7) 或 (29)，则 Δ^{-1} 会成为理论定义的一部分，人们可能会担心规范固定项会明确破坏理论的规范不变性。但实际并非如此，因为 Δ^{-1} 只会出现在我们推导的中间步骤，不会出现在最终作用量与运动方程中。事实上，函数 F_{ij} 在 Δ 处解析，且 Δ^{-1} 出现在 (37) 中只是形式上的。如果追求数学严谨性，我们需要为 (37) 中的 Δ 添加规范固定项，即我们在 (37) 中做如下替换:

$$\Delta \to \Delta + H_{\mathrm{GF}} \tag{38}$$

which is invertible. However, in the EoM and in the action, Δ^{-1} would disappear because it is always multiplied by Δ . Therefore, we can safely take the limit of zero gauge fixing parameters in all the other terms in the Taylor expansion of (37) to finally recover gauge invariance for the action and the EoM.

替换后的算符是可逆的。但在运动方程和作用量中，Δ^{-1} 总会与 Δ 相乘，因此会消失。因此我们可以放心地对 (37) 泰勒展开中所有其他项取规范固定参数趋于零的极限，最终使作用量和运动方程恢复规范不变性。

Appendix B: Propagators

附录 B: 传播子

In order to implement the recipe developed in the main text, we here consider two explicit examples: pure gravity and a general scalar theory. In particular, we will derive the tree-level propagator in both cases.

为了落实正文提出的计算方案，我们在此考虑两个明确的例子: 纯引力和广义标量理论。我们将具体推导出这两种情况下的树级传播子。

Graviton Propagator

引力子传播子

As a first example we consider the purely gravitational theory for which the only non-zero component of the Δ operator is Δ_{11} , but in the flat spacetime background $g_{\mu\nu} = \eta_{\mu\nu}$ and $E_{\mu\nu} = 0$ (Einstein’s EoM), hence

作为第一个例子，我们研究纯引力理论，该理论中 Δ 算符仅存非零分量 Δ_{11} ，而在平直时空背景下满足 $g_{\mu\nu} = \eta_{\mu\nu}$ 和 $E_{\mu\nu} = 0$ (爱因斯坦运动方程)，因此有

$$
\begin{aligned}
\Delta_{\mu\upsilon,\alpha\beta}(y,x) &= \frac{\delta^2 S_\ell}{\delta g^{\mu\upsilon}(y)\,\delta g^{\alpha\beta}(x)} = \frac{2}{\kappa^2}\left[\frac{\delta^2\left(\int \mathrm{d}^D z\sqrt{-g(z)}\mathcal{L}_g(z)\right)}{\delta g^{\mu\upsilon}(y)\,\delta g^{\alpha\beta}(x)}\right]_{\mathbf{g}=\eta} \\
&= \frac{2}{\kappa^2}\left[\frac{\delta}{\delta g^{\mu\upsilon}(y)}\left(\frac{\delta\left(\int \mathrm{d}^D z\sqrt{-g(z)}\mathcal{L}_g(z)\right)}{\delta g^{\alpha\beta}(x)}\right)\right]_{\mathbf{g}=\eta} \\
&= \frac{2}{\kappa^2}\left[\frac{\delta G_{\alpha\beta}(x)}{\delta g^{\mu\upsilon}(y)}\right]_{\mathbf{g}=\eta} \\
&= \frac{2}{\kappa^2}\left[\frac{\delta R_{\alpha\beta}(x)}{\delta g^{\mu\upsilon}(y)} - \frac{1}{2}g_{\alpha\beta}(x)\,g^{\gamma\delta}(x)\,\frac{\delta R_{\gamma\delta}(x)}{\delta g^{\mu\upsilon}(y)}\right]_{\mathbf{g}=\eta} \\
&= \frac{2}{\kappa^2}\frac{1}{2}\left[\frac{1}{2}\left(\eta_{\alpha\mu}\eta_{\beta\upsilon} + \eta_{\beta\mu}\eta_{\alpha\upsilon}\right)\Box_x - \eta_{\mu\upsilon}\eta_{\alpha\beta}\Box_x + \ldots\right]\delta^D(x-y) \\
&= \frac{2}{\kappa^2}\frac{1}{2}\left[P^{(2)}(x) - (D-2)\,P^{(0)}(x)\right]_{\mu\upsilon,\alpha\beta}\Box_x\delta^D(x-y)\,, \\
&\equiv \Delta_{\mu\upsilon,\alpha\beta}(x)\,\delta^D(x-y)\,,
\end{aligned}
\tag{39}
$$

where the ”dots” stay for other second-order derivative terms. The first three steps in (39) are general for any background, while from the fourth step we restricted to the case of the Minkowski spacetime. Furthermore, $P^{(2)}$ and $P^{(0)}$ are the spin-projectors in D -dimensions whose definitions read [15, 24]

其中“省略号”代表其他二阶导数项。(39) 的前三步对任意背景都成立，从第四步开始我们限制在闵氏时空的情况。此外，$P^{(2)}$ 和 $P^{(0)}$ 是 D 维中的自旋投影算符，其定义为 [15, 24]

$$
\begin{aligned}
P^{(2)}_{\mu\upsilon,\rho\sigma}(x) &= \frac{1}{2}\left(\theta_{\mu\rho}\theta_{\upsilon\sigma} + \theta_{\mu\sigma}\theta_{\upsilon\rho}\right) - \frac{1}{D-1}\theta_{\mu\upsilon}\theta_{\rho\sigma}, \\
P^{(0)}_{\mu\nu,\rho\sigma}(x) &= \frac{1}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma} \\
\theta_{\mu\upsilon} = \eta_{\mu\upsilon} - \frac{\partial_\mu\partial_\upsilon}{\Box}&,\ \omega_{\mu\upsilon} = \frac{\partial_\mu\partial_\upsilon}{\Box}.
\end{aligned}
\tag{40}
$$

Notice that the last by one equality in (39) is exact because the projectors reconstruct also the terms shortly indicated with dots.

注意，(39) 中倒数第二个等式是精确的，因为投影算符也可以重建出省略号所代表的项。

Finally, the Δ -operator for the purely gravitational theory reads as follows:

最后，纯引力理论的 Δ 算符形式如下:

$$
\Delta(x,y) = \frac{1}{\kappa^2}\left[P^{(2)}(x) - (D-2)\,P^{(0)}(x)\right]\Box_x\delta^D(y-x)\,, \tag{41}
$$

where we did not displayed the four spacetime indexes.

此处我们没有写出四个时空指标。

In (39) we used the following functional derivatives [11]:

在 (39) 中我们使用了如下泛函导数 [11]:

$$\frac{\delta R_{\alpha\beta}(x)}{\delta g^{\mu\nu}(y)}=\Big[\frac{1}{4}\left(g_{\alpha\mu}g_{\beta\nu}+g_{\beta\mu}g_{\alpha\nu}\right)\Box+\frac{1}{2}g_{\mu\nu}\nabla_{\alpha}\nabla_{\beta}$$

$$-\frac{1}{2}\left(g_{\alpha\mu}\nabla_{\beta}\nabla_{\nu}+g_{\alpha\nu}\nabla_{\beta}\nabla_{\mu}\right)\Big]_{x}\frac{\delta^{D}(x-y)}{\sqrt{-g(y)}},$$

$$g^{\alpha\beta}(x)\frac{\delta R_{\alpha\beta}(x)}{\delta g^{\mu\upsilon}(y)}=\Big[g_{\mu\upsilon}\Box-\frac{1}{2}\left(\nabla_{\mu}\nabla_{\upsilon}+\nabla_{\upsilon}\nabla_{\mu}\right)\Big]_{x}\frac{\delta^{D}(x-y)}{\sqrt{-g(y)}}. \tag{42}$$

We can now expand the purely gravitational theory at the second order in the perturbation $h_{\mu\nu}$ in the Minkowski background, which satisfies the EoM $E_{\mu\nu}=0$, namely

现在我们可以在闵氏背景下将纯引力理论对微扰 $h_{\mu\nu}$ 做二阶展开，该背景满足运动方程 $E_{\mu\nu}=0$ ，即

$$S_{g}[g]=S_{g}[\bar{g}+h]=S_{g}[\bar{g}]+\frac{1}{2}\int d\mu(x_{1})\,d\mu(x_{2})\,h^{\mu\upsilon}(x_{1})\frac{\delta^{2}S_{g}[\bar{g}]}{\delta h^{\mu\upsilon}(x_{1})\,\delta h^{\rho\sigma}(x_{2})}$$

$$\times h^{\rho\sigma}(x_{2})+O\left(h^{3}\right),\ h^{\alpha\beta}\equiv\delta g^{\alpha\beta}, \tag{43}$$

where the second-order term in the above expansion of the action reads

其中上述作用量展开中的二阶项为

$$S_{g}^{(2)}=\int d^{D}x\Big[\sqrt{-g}\frac{2}{\kappa^{2}}R+\sqrt{-g}E_{\alpha\beta}F(\Delta_{\Lambda})^{\alpha\beta,\mu\nu}E_{\mu\nu}\Big]^{(2)}$$

$$=\frac{1}{2}\int d\mu_{y}\int d\mu_{x}h^{\alpha\beta}(y)\left(\frac{\delta^{2}S}{\delta h^{\alpha\beta}(y)\,\delta h^{\mu\upsilon}(x)}\right)h^{\mu\upsilon}(x)$$

$$=\frac{1}{2}\int d\mu_{y}\int d\mu_{x}h^{\alpha\beta}(y)\left(\frac{\delta^{2}S_{\ell}}{\delta h^{\alpha\beta}(y)\,\delta h^{\mu\upsilon}(x)}\right)h^{\mu\upsilon}(x)+\frac{1}{2}\int d\mu_{y}\int d\mu_{x}h^{\gamma\delta}$$

$$(y)\times\left[\frac{\delta}{\delta h^{\gamma\delta}(y)}\frac{\delta}{\delta h^{\rho\sigma}(x)}\left(\int d\mu_{z}E_{\alpha\beta}(z)\,F(\Delta_{\Lambda}(z))^{\alpha\beta,\mu\nu}E_{\mu\nu}(z)\right)\right]h^{\rho\sigma}(x)$$

$$=\frac{1}{2}\int_{x}\int_{y}\Big[h^{\alpha\beta}(y)\left(\frac{\delta^{2}S_{\ell}}{\delta h^{\alpha\beta}(y)\,\delta h^{\mu\upsilon}(x)}\right)h^{\mu\upsilon}(x)$$

$$+2h^{\gamma\delta}(y)\left(\int_{z}\frac{\delta E_{\alpha\beta}(z)}{\delta h^{\gamma\delta}(y)}F(\Delta_{\Lambda}(z))^{\alpha\beta,\mu\upsilon}\frac{\delta E_{\mu\upsilon}(z)}{\delta h^{\rho\sigma}(x)}\right)h^{\rho\sigma}(x)],$$

$$=\frac{1}{2}\int_{x}\int_{y}\Big[h^{\alpha\beta}(y)\left(\frac{\delta^{2}S_{\ell}}{\delta h^{\alpha\beta}(y)\,\delta h^{\mu\upsilon}(x)}\right)h^{\mu\upsilon}(x)$$

$$+h^{\gamma\delta}(y)\left(\int_{z}2\frac{\delta E_{\alpha\beta}(z)}{\delta h^{\gamma\delta}(y)}F(\Delta_{\Lambda}(z))^{\alpha\beta,\mu\nu}\Delta(x,z)_{\rho\sigma,\mu\nu}\right)h^{\rho\sigma}(x)\Big].$$

The overall 1/2 factor is due to the functional expansion at the second order, while the multiplicative factor 2 in the second term at the forth step comes from the symmetric property of Δ , namely $\Delta(x,z)_{\rho\sigma,\mu\nu} = \Delta(z,x)_{\mu\nu,\rho\sigma}$. Using again the latter property,

整体 1/2 因子来自二阶泛函展开, 而第四步中第二项的乘积因子 2 来自 Δ 的对称性, 即 $\Delta(x,z)_{\rho\sigma,\mu\nu} = \Delta(z,x)_{\mu\nu,\rho\sigma}$。再次利用该性质,

$$S_{\mathrm{g}}^{(2)} = \frac{1}{2}\int d\mu\,(y)\int d\mu\,(x)\Big[h^{\alpha\beta}\,(y)\Big(\frac{\delta^2 S_\ell}{\delta h^{\alpha\beta}\,(y)\,\delta h^{\mu\upsilon}\,(x)}\Big)h^{\mu\upsilon}\,(x)$$

$$+h^{\gamma\delta}\,(y)\Big(\int d\mu\,(z)\,2\frac{\delta E_{\alpha\beta}\,(z)}{\delta h^{\gamma\delta}\,(y)}F(\Delta_\Lambda\,(z))^{\alpha\beta,\mu\upsilon}\Delta(z,x)_{\mu\nu,\rho\sigma}\Big)h^{\rho\sigma}\,(x)\Big]$$

$$= \frac{1}{2}\int d\mu\,(y)\int d\mu\,(x)\Big\{h^{\alpha\beta}\,(y)\,\frac{\delta E_{\mu\nu}\,(x)}{\delta h^{\alpha\beta}\,(y)}h^{\mu\upsilon}\,(x)$$

$$+h^{\gamma\delta}\,(y)\Big[\int d\mu\,(z)\,2\frac{\delta E_{\alpha\beta}\,(z)}{\delta h^{\gamma\delta}\,(y)}F(\Delta_\Lambda\,(z))^{\alpha\beta,\mu\nu}\Delta(z,x)_{\mu\nu,\rho\sigma}\Big]h^{\rho\sigma}\,(x)\Big\}.$$

Changing the name of the indexes in the second quadratic term in $h_{\mu\nu}$, we get

对 $h_{\mu\nu}$ 中第二个二次项做指标重命名后，我们得到

$$S_{\mathrm{g}}^{(2)} = \frac{1}{2}\int d\mu_y\int d\mu_x\Big\{h^{\alpha\beta}\,(y)\,\frac{\delta E_{\mu\upsilon}\,(x)}{\delta h^{\alpha\beta}\,(y)}h^{\mu\upsilon}\,(x)$$

$$+h^{\gamma\delta}\,(y)\Bigg[\int d\mu_w\int d\mu_z\,\frac{\delta E_{\alpha\beta}\,(w)}{\delta h^{\gamma\delta}\,(y)}\frac{\Big[2F(w,z)^{\alpha\beta,\mu\nu}\Delta(z,x)_{\mu\nu,\rho\sigma}\Big]}{\Big[2F(w,z)^{\alpha\beta+\rho}\Big]}h^{\rho\sigma}\,(x)\Bigg].\Bigg\}$$

Now we replace the product in the box, i.e., $2F\,(\Delta)\,\Delta$, with (7),

现在我们将框内的乘积，即 $2F\,(\Delta)\,\Delta$ ，替换为 (7),

$$S_{\mathrm{g}}^{(2)} = \frac{1}{2}\int_y\int_x\Big\{h^{\alpha\beta}\,(y)\,\frac{\delta E_{\mu\upsilon}\,(x)}{\delta h^{\alpha\beta}\,(y)}h^{\mu\upsilon}\,(x)$$

$$+h^{\gamma\delta}\,(y)\Big[\int_w\frac{\delta E_{\alpha\beta}\,(w)}{\delta h^{\gamma\delta}\,(y)}\left(e^{H(\Delta)}-1\right)(w,x)^{\alpha\beta}{}_{\rho\sigma}\Big]h^{\rho\sigma}\,(x)\Big\}$$

$$= \frac{1}{2}\int_y\int_x\Big\{h^{\alpha\beta}\,(y)\,\frac{\delta E_{\mu\nu}\,(x)}{\delta h^{\alpha\beta}\,(y)}h^{\mu\nu}\,(x)$$

$$+h^{\gamma\delta}\,(y)\Bigg[\int_w\frac{\delta E_{\alpha\beta}\,(w)}{\delta h^{\gamma\delta}\,(y)}\Bigg(e^{H(\Delta)}(w,x)^{\alpha\beta}{}_{\rho\sigma}-\frac{\delta(w,x)^{\delta\alpha\beta}{}_{\rho\sigma}}{\sqrt{-\mathrm{g}\,(x)}}\Bigg)h^{\rho\sigma}\,(x)\Bigg]$$

$$= \frac{1}{2}\int_y\int_x\big\{h^{\alpha\beta}\,(y)\,\Delta(y,x)_{\alpha\beta,\mu\nu}h^{\mu\nu}\,(x)$$

$$+h^{\gamma\delta}\,(y)\Bigg[\int_w\Delta(y,w)_{\gamma\delta,\alpha\beta}\Bigg(e^{H(\Delta)}(w,x)^{\alpha\beta}{}_{\rho\sigma}-\frac{\delta\,(w,x)\,\delta^{\alpha\beta}{}_{\rho\sigma}}{\sqrt{-\mathrm{g}\,(x)}}\Bigg)h^{\rho\sigma}\,(x)\Bigg]\Bigg\}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_y\int_x\int_w \left\{h^{\gamma\delta}(y)\,\frac{\delta E_{\alpha\beta}(w)}{\delta h^{\gamma\delta}(y)}\left(e^{H(\Delta)}(w,x)^{\alpha\beta}{}_{\rho\sigma}\right)h^{\rho\sigma}(x)\right\}\\
&= \frac{1}{2}\int d^Dx\left[h^{\alpha\beta}\left(\frac{\delta E_{\gamma\delta}}{\delta h^{\alpha\beta}}\right)\left[e^{H(\Delta_\Lambda)}\right]^{\gamma\delta}_{\mu\nu}h^{\mu\nu}\right]\\
&= \frac{1}{2}\int d^Dx\left[h^{\alpha\beta}\left(\frac{\delta^2 S_\ell}{\delta h^{\alpha\beta}\delta h^{\gamma\delta}}\right)\left[e^{H(\Delta_\Lambda)}\right]^{\gamma\delta}_{\mu\nu}h^{\mu\nu}\right]\\
&= \frac{1}{2\kappa^2}\int d^Dx\left\{h^{\alpha\beta}\left[\left(P^{(2)}-(D-2)\,P^{(0)}\right)\right]_{\alpha\beta,\gamma\delta}\left[e^{H(\Delta_\Lambda)}\right]^{\gamma\delta}_{\mu\nu}h^{\mu\nu}\right\}.
\end{aligned}
$$
(44)

In the last three steps we replaced $\sqrt{-g} = 1$ because we are expanding around the Minkowski background. In the last step of (44), we used the result (39). Finally, replacing the argument of the form factor $\exp H(\Delta_\Lambda)$ in (44) with (41), we get

在最后三步中我们替换了 $\sqrt{-g} = 1$，这是因为我们围绕闵氏背景展开。在 (44) 的最后一步，我们使用了结果 (39)。最后，将 (44) 中形状因子 $\exp H(\Delta_\Lambda)$ 的自变量替换为 (41)，我们得到

$$
\begin{aligned}
S_{\mathrm{g}}^{(2)} &= \frac{1}{2\kappa^2}\int d^Dx\\
&\times\left\{h^{\alpha\beta}\left[\left(P^{(2)}-(D-2)\,P^{(0)}\right)\Box\right]_{\alpha\beta,\gamma\delta}\left[e^{H\left(\frac{1}{\kappa^2}\frac{P^{(2)}-(D-2)P^{(0)}}{\Lambda^4}\Box\right)}\right]^{\gamma\delta}_{\mu\nu}h^{\mu\nu}\right\}\\
&= \frac{1}{2\kappa^2}\int d^Dx\\
&\times\left\{h^{\alpha\beta}\left[\left(P^{(2)}e^{H\left(\frac{\Box}{\kappa^2\Lambda^4}\right)}-(D-2)\,P^{(0)}e^{H\left(\frac{-(D-2)\Box}{\kappa^2\Lambda^4}\right)}\right)\Box\right]_{\alpha\beta,\mu\nu}h^{\mu\nu}\right\}\\
&\equiv \frac{1}{2}\int d^Dx\,h^{\alpha\beta}\mathcal{O}_{\alpha\beta,\mu\nu}h^{\mu\nu},
\end{aligned}
\tag{45}
$$

from which after introducing the gauge fixing it is obtained the following gauge-independent part of the graviton propagator [15]:

由此，在引入规范固定后，我们得到引力子传播子如下的规范不变部分 [15]:

$$
\mathcal{O}^{-1} = \kappa^2\left[\frac{P^{(2)}}{\Box e^{H\left(\frac{\Box}{\kappa^2\Lambda^4}\right)}} - \frac{P^{(0)}}{\Box\,(D-2)\,e^{H\left(\frac{-(D-2)\Box}{\kappa^2\Lambda^4}\right)}}\right].
\tag{46}
$$

The tree-level unitarity is guaranteed whether the asymptotically polynomial entire function $H(z)$ is such that $H(z) = 0$. Moreover, $H(z) = H(-z)$ in order to ensure the super-renormalizability of the theory.

只要渐近多项式整函数 $H(z)$ 满足 $H(z) = 0$，树级幺正性就得到保证。此外，为了保证理论的超可重整性，要求 $H(z) = H(-z)$。

Free and Interacting Scalar Fields

自由标量场与相互作用标量场

For a free scalar field, the nonlocal Lagrangian, the local EoM E_m , and the Δ_{22} operator are

对于自由标量场，非局域拉格朗日量、局域运动方程 E_m 以及 Δ_{22} 算符为

$$\mathcal{L}_m = \frac{1}{2}\phi\Box\phi + E_m F(\Delta_\Lambda)_{22} E_m$$

$$E_m = \Box\phi \tag{47}$$

$$\Delta_{22} = \frac{\delta E_m}{\delta\phi} = \Box. \tag{48}$$

Replacing E_m and Δ_{22} in $\mathcal{L}_m$, we end up with the following nonlocal Lagrangian:

将 E_m 和 Δ_{22} 代入 $\mathcal{L}_m$ 后，我们得到如下非局域拉格朗日量:

$$\mathcal{L}_m = \frac{1}{2}\phi\Box\phi + (\Box\phi)\,\frac{e^{H(\Box)} - 1}{2\Box}\,(\Box\phi) = \frac{1}{2}\phi\Box e^{H(\Box)}\phi. \tag{49}$$

Therefore, the propagator is proportional to

因此，传播子正比于

(50)

$$\frac{e^{-H(\Box)}}{\Box}.$$

For an interacting scalar field, whose local Lagrangian and EoM read

对于相互作用标量场，其局域拉格朗日量和运动方程为

$$\mathcal{L}_m^{(\mathrm{loc})} = \frac{1}{2}\phi\Box\phi - V(\phi)\,, \tag{51}$$

$$E_m = \Box\phi - V'(\phi)\,, \tag{52}$$

the nonlocal theory is

非局域理论为

$$\mathcal{L}_m = \frac{1}{2}\phi\Box\phi - V(\phi) + (\Box\phi - V'(\phi))F(\Delta)_{22}(\Box\phi - V'(\phi)),$$

$$\Delta_{22} = \Box - V''(\phi)$$

$$F(\Delta)_{22} = \frac{e^{H(\Box - V''(\phi))} - 1}{2(\Box - V''(\phi))}. \tag{53}$$

If we switch off the interaction, the Lagrangian (53) turns into (49).

如果关闭相互作用，拉格朗日量 (53) 将退化为 (49)。

Appendix C: Symmetry Properties of the Hessian

附录 C: 海森矩阵的对称性

In this section we explicitly prove that the Hessian operator Δ is symmetric, namely

在本节中，我们明确证明海森算符 Δ 是对称的，即

$$\int dx \int dy A_i(x)\Delta_{ij}(x,y)B_j(y) = \int dx \int dy B_j(y)\Delta_{ji}(y,x)A_i(x) \tag{54}$$

In order to prove the above statement it is sufficient to consider the following quite general contribution to the action:

为证明上述命题，只需考虑作用量的如下十分一般的贡献项:

$$S = \int dx \Phi_1(x)(\partial_x^n \Phi_2(x))\Phi_3(x), \tag{55}$$

where $\Phi_1(x), \Phi_2(x)$, and $\Phi_3(x)$ are three general fields. For the sake of simplicity, we assume $\Phi_3(x)$ to be an external classical field, and we compute the components of Hessian's operator, which is a 2×2 matrix in the space of the fields $\Phi_1(x)$ and $\Phi_2(x)$

其中 $\Phi_1(x), \Phi_2(x)$ 、$\Phi_3(x)$ 是三个任意场。为简化起见，我们假设 $\Phi_3(x)$ 是外部经典场，随后计算海森算符的分量，该算符在场 $\Phi_1(x)$ 与 $\Phi_2(x)$ 的空间中是一个 2×2 矩阵

$$\frac{\delta S}{\delta \Phi_1(y)} = \int dx \delta(x-y)(\partial_x^n \Phi_2(x))\Phi_3(x), \tag{56}$$

$$\Delta_{21}(z,y) = \frac{\delta^2 S}{\delta\Phi_2(z)\delta\Phi_1(y)} = (-1)^n \int dx \delta(x-z)\partial_x^n(\delta(x-y)\Phi_3(x))$$

$$= (-1)^n \partial_z^n(\delta(z-y)\Phi_3(z)), \tag{57}$$

$$\frac{\delta S}{\delta \Phi_2(y)} = (-1)^n \int dx \delta(x-y)\partial_x^n(\Phi_1(x)\Phi_3(x)), \tag{58}$$

$$\Delta_{12}(z,y) = \frac{\delta^2 S}{\delta\Phi_1(z)\,\delta\Phi_2(y)} = (-1)^n(-1)^n \int dx \delta(x-z)\,\Phi_3(x)\,\partial_x^n(\delta(x-y))$$

$$= \Phi_3(z)\,\partial_z^n(\delta(z-y)). \tag{59}$$

In deriving the components of the Hessian, we integrated by parts several times. Moreover, the Hessian operator has zero diagonal elements because the action (55) is linear in all fields.

在推导海森矩阵分量的过程中，我们多次使用分部积分。此外，由于作用量 (55) 对所有场都是线性的，因此海森算符的对角元为零。

Given two general fields $A(z)$ and $B(y)$, we now prove the following identity:

给定两个任意场 $A(z)$ 和 $B(y)$ ，我们现在证明如下恒等式:

$$\int dz \int dy A(z)\,\Delta_{21}(z,y)\,B(y) = \int dz \int dy B(y)\,\Delta_{12}(y,z)\,A(z). \tag{60}$$

Replacing (57) in the left-hand side of (60), we find

将 (57) 代入 (60) 的左侧，我们得到

$$\int dz \int dy A(z)\,\Delta_{21}(z,y)\,B(y) = \int dz \int dy A(z)\left[(-1)^n \partial_z^n(\delta(z-y)\,\Phi_3(z))\right] B(y)$$

$$= (-1)^n \int dz \int dy (-1)^n\,(\partial_z^n A(z))\,\delta(z-y)\,\Phi_3(z)\,B(y)$$

$$= \int dy \left(\partial_y^n A(y)\right)\Phi_3(y)\,B(y)$$

$$= \int dy (-1)^n A(y)\,\partial_y^n(\Phi_3(y)\,B(y)). \tag{61}$$

Similarly, we replace (59) in the right-hand side of (60)

类似地，我们将 (59) 代入 (60) 的右侧

$$\int dz \int dy B(y)\,\Delta_{12}(y,z)\,A(z) = \int dz \int dy B(y)\left[\Phi_3(y)\,\partial_y^n(\delta(y-z))\right] A(z) \tag{62}$$

$$= \int dz \int dy (-1)^n \partial_y^n(B(y)\,\Phi_3(y))\,\delta(y-z)\,A(z)$$

$$= \int dy (-1)^n A(y)\,\partial_y^n(\Phi_3(y)\,B(y)) = (61). \tag{63}$$

Hence, we have proved (60).

由此，我们完成了对 (60) 的证明。

We can also swap the fields $A(z)$ and $B(y)$ in (62), and the result does not change

我们也可以交换 (62) 中的场 $A(z)$ 和 $B(y)$，结果不发生改变

$$\int dz \int dy A(z) \Delta_{12}(y,z) B(y) = \int dz \int dy A(z) \left[\Phi_3(y) \partial_y^n (\delta(y-z))\right] B(y) \tag{64}$$

$$= \int dz \int dy (-1)^n \partial_y^n (\Phi_3(y) B(y)) \delta(y-z) A(z)$$

$$= \int dy (-1)^n A(y) \partial_y^n (\Phi_3(y) B(y)) = (61), \tag{65}$$

which also proves the statement in the text below formula (24).

这也证明了式 (24) 下方正文中的命题。

Appendix D: Proof of the Theorem in the Main Text

附录 D: 正文中定理的证明

In order to prove the theorem about the linear and nonlinear stability of the nonlocal theory (1), we have to expand perturbatively in ε (see (11)) the EoM (8)

为了证明非局部理论 (1) 的线性与非线性稳定性定理，我们需要对运动方程 (8) 围绕 ε 做微扰展开 (见式 (11))

$$\mathcal{E} = \mathbf{e}^{\mathbf{H}(\Delta_\Lambda)} \mathbf{E} + O\left(\mathbf{E}^2\right) = 0. \tag{66}$$

Since we want to study the stability of exact solutions of Einstein's theory coupled to matter, we assume to expand around an exact solution consistent with $\mathbf{E}^{(0)} = 0$ (12) (for the sake of simplicity, we use the bold notation in place of the Latin indexes for the fields).

由于我们要研究耦合物质的爱因斯坦理论精确解的稳定性，我们假设围绕满足 $\mathbf{E}^{(0)} = 0$ (12) 的精确解展开 (为简便起见，我们用粗体记号代替场的拉丁指标)。

Hence, at the zero order in ε , i.e., ε^0 , we have

因此，在 ε 的零阶，即 ε^0 ，我们得到

$$\mathbf{e}^{\mathbf{H}^{(0)}(\Delta_\Lambda)} \mathbf{E}^{(0)} + O\left(\mathbf{E}^{(0)2}\right) = 0, \tag{67}$$

which is satisfied because by hypothesis $\mathbf{E}^{(0)} = 0$.

根据假设 $\mathbf{E}^{(0)} = 0$ ，上式成立。

At the first order ε^1 , we get

在 ε^1 的一阶，我们得到

$$\mathbf{e}^{\mathbf{H}^{(1)}(\Delta_\Lambda)}\mathbf{E}^{(\mathbf{0})}+\mathbf{e}^{\mathbf{H}^{(0)}(\Delta_\Lambda)}\mathbf{E}^{(\mathbf{1})}+O\left(\mathbf{E}^{(\mathbf{0})}\mathbf{E}^{(\mathbf{1})}\right)=0\ \Rightarrow\ \mathbf{E}^{(\mathbf{1})}=0, \tag{68}$$

where we used $\mathbf{E}^{(\mathbf{0})}=0$.

此处我们用到了 $\mathbf{E}^{(\mathbf{0})}=0$ 。

At the second order ε^2 , we get:

在 ε^2 的二阶，我们得到:

$$\mathbf{e}^{\mathbf{H}^{(2)}(\Delta_\Lambda)}\mathbf{E}^{(\mathbf{0})}+\mathbf{e}^{\mathbf{H}^{(1)}(\Delta_\Lambda)}\mathbf{E}^{(1)}+\mathbf{e}^{\mathbf{H}^{(\mathbf{0})}(\Delta_\Lambda)}\mathbf{E}^{(2)}+O\left(\mathbf{E}^{(1)}E^{(1)}\right)$$

$$+O\left(\mathbf{E}^{(2)}E^{(0)}\right)=0\Rightarrow\mathbf{E}^{(2)}=0, \tag{69}$$

where we used $\mathbf{E}^{(\mathbf{0})}=0$ and $\mathbf{E}^{(\mathbf{1})}=0$.

此处我们用到了 $\mathbf{E}^{(\mathbf{0})}=0$ 和 $\mathbf{E}^{(\mathbf{1})}=0$ 。

Finally, at the order ε^n ,

最后，在 ε^n 阶，

$$\mathbf{e}^{\mathbf{H}^{(\mathbf{n})}(\Delta_\Lambda)}\mathbf{E}^{(\mathbf{0})}+\mathbf{e}^{\mathbf{H}^{(\mathbf{n-1})}(\Delta_\Lambda)}\mathbf{E}^{(\mathbf{1})}+\mathbf{e}^{\mathbf{H}^{(\mathbf{n-2})}(\Delta_\Lambda)}\mathbf{E}^{(\mathbf{2})}+\cdots+\mathbf{e}^{\mathbf{H}^{(\mathbf{0})}(\Delta_\Lambda)}\mathbf{E}^{(\mathbf{n})}+$$

$$+O\left(\mathbf{E}^{(\mathbf{n})}\mathbf{E}^{(\mathbf{0})}\right)+O\left(\mathbf{E}^{(\mathbf{n-1})}\mathbf{E}^{(\mathbf{1})}\right)+\cdots+O\left(\mathbf{E}^{(\mathbf{1})}\mathbf{E}^{(\mathbf{n-1})}\right)$$

$$+O\left(\mathbf{E}^{(\mathbf{0})}E^{(\mathbf{n})}\right)=0\ \Rightarrow\ \mathbf{E}^{(\mathbf{n})}=0, \tag{70}$$

where we used $\mathbf{E}^{(\mathbf{0})}=0,\mathbf{E}^{(\mathbf{1})}=0,\ldots,\mathbf{E}^{(\mathbf{n-1})}=0$.

此处我们用到了 $\mathbf{E}^{(\mathbf{0})}=0,\mathbf{E}^{(\mathbf{1})}=0,\ldots,\mathbf{E}^{(\mathbf{n-1})}=0$ 。

Therefore,

因此，

$$\mathcal{E}^{(\mathbf{n})}=0\Rightarrow\mathbf{E}^{(\mathbf{n})}=0. \tag{71}$$

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